ON THE TEMPERATURE OR CONCENTRATION FIELDS PRODUCED INSIDE AN INFINITE OR FINITE DOMAIN BY MOVING SURFACES AT WHICH THE TEMPERATURE OR CONCENTRATION ARE GIVEN AS FUNCTIONS OF TIME

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We show that one or more special transformations of the general equation of heat conduction (diffusion) enable us to use the latter to solve a whole series of problems of the type indicated in the title. In particular, we solve the problem where the temperature (concentration) field is produced by a uniform or uniformly varying motion of a plane at which the temperature (concentration) is given as a function of time, as well as analogous problems for axially semi-infinite prismatic or cylindrical rods, etc., obtaining relatively simple linear integral equations for solving the problems in some of the more general cases.

In [1] we determined the temperature field ahead of the front of a heat source moving in an unbounded isotropic medium under the assumption that the heat source temperature is either constant or a given function of time (the one-dimensional problem).

The method used in [1] was based on consideration of the process in a coordinate system moving together with the heat source. This enabled us to obtain the solution of the problem directly in quadratures in the case of uniform motion of the front and to reduce the matter in the case of nonuniform motion to the solution of a certain integral equation which yields the required temperature field even when the coordinate of the front increases in proportion to the square root of the time.

We shall show that if, in addition to introducing the above coordinate system, we subject the heat condition equation in this system to a certain special transformation, then the latter assumes a form which enables us to solve the indicated problem in terms of known functions not only in the above cases, but also when the front accelerates or decelerates uniformly and when not only the initial state of the system and distribution of the specified heat sources within it, but also the time dependence of the temperature of the front can be specified in arbitrary fashion. The resulting exact solution can therefore be used for approximate investigation of the general problem in the case where the front moves according to a more complex law but admits of sufficiently accurate step-by-step approximation by a uniformly varying motion. The same purpose is served by the other exact solution which we obtain in the present paper, namely the solution for the case where the front $x = \xi$ moves according to the law $\xi = \sqrt{A + Bt + Ct^2}$, where A, B and C are arbitrary constants. At the same time, the new form of the basic equation of the problem enables us to reduce solution of the latter in the general case to certain nonconventional integral equations which in some cases provide a more effective and convenient pathway to the solution. We also note that considerations similar to those of [2, 3] and our own method combined with the results obtained in these studies make it possible to solve many two- and three-dimensional problems of the theory of heat conduction and diffusion with moving boundaries. This includes the problem of a semi-infinite prismatic rod whose end $x = R(t)$ moves uniformly, with a uniformly varying velocity, or according to the law $R(t) = (A + Bt + Ct^2)^{1/2}$, and whose side faces move along the coordinate axes according to the laws $R_i(t) = (M_i t^2 + N_i t + P_i)^{1/2}$, where i is the index of the corresponding axis and M_i, N_i, P_i are arbitrary constants which depend on the index i.

The same procedure can be used to solve the corresponding problems for a semi-infinite cylinder whose side-surface radius varies according to the law $R(t)=(Mt^2+Nt+P)^{1/2}$, and whose endface either accelerates or decelerates uniformly, as well as various other problems, some of which are mentioned in Sects. 4 and 5 of the present paper.

1. Let us consider the above problem, which reduces in the one-dimensional case to solution of the equation

$$a\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + f(x, t), \quad a = \text{const} \tag{1.1}$$

for a semi-infinite domain $x > R(t)$, where $R(t)$ is some function of time, and where we know the initial state

$$u|_{t=0} = F(x) \quad \text{for } x > R(0), \quad u|_{x=R(t)} = \varphi(t)$$

at the moving boundary.

Here $f(x, t)$ is a given function of its arguments.

In addition, the required solution must usually satisfy the requirement of boundedness or vanishing at infinity; at the very least the character of its growth at infinity must be indicated.

Without limiting generality we can clearly assume that $R(0) = 0$ and $a = 1$. This we shall do below. We can also set $F(x) \equiv 0$. In fact, let us suppose that this is not the case in the initial problem for u. Then, subtracting, for example, the function (*)

$$u_1(x, t) = \frac{1}{2\sqrt{\pi a t}} \int_0^\infty F(\alpha) \exp\frac{-(\alpha - x)^2}{4at}\, d\alpha \tag{1.2}$$

satisfying the equations

$$a\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_1}{\partial t}, \quad u_1|_{t=0} = F(x) \quad (x > 0) \tag{1.3}$$

from u, we find that the difference function $U = u - u_1$ satisfies the same Eq. (1.1) as u, but under the initial condition $U|_{t=0} = 0$. The condition for $x = R(t)$ for this function is of the form

$$U|_{x=R(t)} = (u - u_1)|_{x=R(t)} = \varphi(t) - u_1[R(t), t] \equiv \bar{\varphi}(t) \tag{1.4}$$

i.e. since $u_1[R(t), t]$ is a known function of t, it follows that the known function $\varphi(t)$ in the condition at the moving boundary must be replaced by another (also known) function $\bar{\varphi}(t)$. This enables us, without limiting the generality of our solution, to set $F(x) \equiv 0$, assuming, if need be, that u is the same as the function U.

Now, setting $\xi = x - R(t)$, i.e. introducing a coordinate system which moves together with the boundary, we obtain the following equation for u:

$$\frac{\partial^2 u}{\partial \xi^2} + R'\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial t} = f(\xi + R, t) \equiv f^*, \quad R' = \frac{dR}{dt} \tag{1.5}$$

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*) We can also take a more general solution of Eq. (1.3) as our u_1, namely

$$u_2 = \frac{1}{2\sqrt{\pi a t}} \int_{-\infty}^{\infty} \Phi(\alpha) \exp\frac{-(\alpha - x)^2}{4at}\, d\alpha$$

where $\Phi(\alpha)$ coincides with $F(\alpha)$ for $\alpha > 0$, but can be chosen arbitrarily for $\alpha < 0$.

Here the derivative with respect to t must be taken for a constant ξ, and the condition at the left-hand boundary is $u|_{\xi=0} = \varphi(t)$.

In the case of uniform motion of the boundary we have $R' = \text{const}$; Eq. (1.5) is solvable directly in this case (see [1]).

In the general case of an arbitrary $R(t)$ we replace the u in (1.5) by the new function v by way of the relation

$$u = qv, \quad q = \exp[-{}^1/_2(R'\xi + {}^1/_2 \textstyle\int R'^2 dt)] \qquad (1.6)$$

Equation (1.5) and the corresponding boundary condition for $\xi = 0$ and the initial condition become

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{R''}{2}\xi v - \frac{\partial v}{\partial t} = \frac{f^*}{q}, \quad R'' = \frac{d^2 R}{dt^2} \qquad (1.7)$$

$$v\Big|_{\xi=0} = \frac{u}{q}\Big|_{\xi=0} = \varphi(t)\exp\Big[\frac{1}{4}\int R'^2 dt\Big], \quad v|_{t=0} = 0 \qquad (1.8)$$

The condition for v as $\xi \to \infty$ follows from the corresponding equation for u.

Equation (1.7) implies, first, that in the case of uniform motion of the boundary ($R'' = 0$) the solution of the problem with a moving boundary reduces simply to the solution of the analogous problem with a fixed boundary. But Eq. (1.7) also implies that the homogeneous equation with results from (1.7) for $f^* = 0$ is amenable to separation of variables even in the case $R'' = \text{const} \neq 0$, i.e. in the case of uniformly varying motion. This makes it possible to solve the general problem formulated by Eqs. (1.7), (1.8) in familiar functions which have already been investigated in detail (as we shall presently show).

2. Let $R = {}^1/_2\,\alpha t^2 + \beta t$, where $\alpha \neq 0$ and β are constants. Then $R'' = \alpha$ and Eq. (1.7) becomes

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{1}{2}\alpha\xi v - \frac{\partial v}{\partial t} = \frac{f^*}{q} \qquad (2.1)$$

Here, by (1.6),

$$q = \exp\Big[-\frac{(\alpha t + \beta)\xi}{2} - \frac{\alpha^2 t^3 + 3\alpha\beta t^2 + 3\beta^2 t}{12}\Big] \qquad (2.2)$$

In this case (1.8) yields

$$v|_{\xi=0} = \varphi(t)\exp[{}^1/_{12}(\alpha^2 t^3 + 3\alpha\beta t^2 + 3\beta^2 t)], \quad v|_{t=0} = 0 \quad (\xi > 0) \qquad (2.3)$$

The homogeneous equation which follows from (2.1) for $f^* = 0$, i.e.

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\alpha}{2}\xi w - \frac{\partial w}{\partial t} = 0 \qquad (2.4)$$

has solutions of the form $w = e^{{}^1/_2 \alpha\lambda t}\mu(\xi)$, $\lambda = \text{const}$, where

$$d^2\mu / d\xi^2 + {}^1/_2\alpha(\xi - \lambda)\mu = 0 \qquad (2.5)$$

This equation is integrable in Bessel functions of order ${}^1/_3$. Let us consider the cases $\alpha > 0$ and $\alpha < 0$ beginning with the latter. Setting $\alpha = -\gamma$, $\gamma > 0$, and

$$\eta = ({}^1/_2\gamma)^{1/3}(\xi - \lambda) \qquad (2.6)$$

where the value $\xi = \infty$ corresponds to $\eta = \infty$, we can rewrite Eq. (2.5) as

$$d^2\mu / d\eta^2 - \eta\mu = 0 \qquad (2.7)$$

The independent particular solutions of Eq. (2.7) are the functions

$$\mu_1(\eta) = \sqrt{\eta}\, I_{1/3}({}^2/_3\eta^{3/2}), \quad \mu_2(\eta) = \sqrt{\eta}\, K_{1/3}({}^2/_3\eta^{3/2}) \qquad (2.8)$$

i.e. the modified Bessel functions of order ${}^1/_3$ of the indicated argument multiplied by $\sqrt{\eta}$. We note that despite the ostensible presence of the square root of η in these formulas, the functions $\mu_1(\eta)$ and $\mu_2(\eta)$ in fact constitute entire functions of η. This is evident from the fact that they are linear combinations of the two other independent particular solutions of Eq. (2.7), namely

$$\mu_3(\eta) = \eta + \frac{\eta^4}{3\cdot 4} + \frac{\eta^7}{3\cdot 4\cdot 6\cdot 7} + \ldots, \quad \mu_4(\eta) = 1 + \frac{\eta^3}{2\cdot 3} + \frac{\eta^6}{2\cdot 3\cdot 5\cdot 6} + \ldots \quad (2.9)$$

which is readily verifiable by substituting them into (2.7).

The eigenfunctions of the problem under consideration are the solutions (e. g. see [4], Sects. 15 and 21 or [5]) of Eq. (2.5) which vanish for $\xi = 0$ and $\xi = \infty$. The latter condition is satisfied by the function $\mu_2(\eta)$, which is given by

$$\mu(\xi) = \sqrt{\xi - \lambda}\, K_{1/3}[{}^1/_3 \sqrt{2\gamma}(\xi - \lambda)^{3/2}] \quad (2.10)$$

to within a constant factor with no special significance.

Let us make use of the familiar formula

$$K_\nu(z) = \frac{\pi}{2\sin \pi\nu}[I_{-\nu}(z) - I_\nu(z)] \quad (2.11)$$

Setting $\nu = {}^1/_3$ in this equation, we readily obtain (see monograph [6], Sect. 4.12 for details) the following expression for the function $\mu(\xi)$ for $\xi < \lambda$ in terms of functions of a real argument:

$$\mu(\xi) = \frac{\pi\sqrt{\lambda - \xi}}{\sqrt{3}}\{J_{1/3}[{}^1/_3\sqrt{2\gamma}(\lambda - \xi)^{3/2}] + J_{-1/3}[{}^1/_3\sqrt{2\gamma}(\lambda - \xi)^{3/2}]\} \quad (2.12)$$

The condition $\mu(0) = 0$ gives us the equation

$$J_{1/3}[{}^1/_3\sqrt{2\gamma}\lambda_n^{3/2}] + J_{-1/3}[{}^1/_3\sqrt{2\gamma}\lambda_n^{3/2}] = 0 \quad (2.13)$$

for finding the eigenvalues λ_n. There are detailed tables (e. g. see [7], p. 103)) of the roots of the equation

$$J_{1/3}(z) + J_{-1/3}(z) = 0 \quad (2.14)$$

There are also tables of functions which differ from (2.10) and (2.12) by a numerical factor only (they are Airy functions; see [8] for information concerning suitable tables); hence, denoting the roots of Eq. (2.14) by z_n, we obtain

$$\lambda_n = ({}^9/_2 z_n^2/\gamma)^{1/3}, \qquad n = 1, 2, 3, \ldots, \infty \quad (2.15)$$

Substituting these values into (2.10) and (2.12), we obtain the eigenfunctions $\mu_n(\xi)$ of the corresponding boundary value problem for Eq. (2.5) which we can now rewrite as

$$d^2\mu_n / d\xi^2 - {}^1/_2\,\gamma(\xi - \lambda_n)\mu_n = 0 \qquad (n = 1, 2, 3, \ldots) \quad (2.16)$$

The functions $\mu_n(\xi)$ form a complete system orthogonal in the interval $(0, \infty)$. In order to make it normalized as well, we need merely recall the formula

$$\int_0^\infty \mu_n^2 d\xi = \frac{2}{\gamma}\left(\frac{d\mu_n}{d\xi}\right)^2\bigg|_{\xi=0} \quad (2.17)$$

and the fact that the derivative occurring in this expression is given by

$$d\mu_n / d\xi|_{\xi=0} = -\sqrt{{}^1/_2\gamma}\,\lambda_n[J_{-2/3}(z_n) - J_{2/3}(z_n)]$$

Hence, the normalized eigenfunctions are of the form

$$\psi_n(\xi) = -\frac{\mu_n(\xi)}{\lambda_n[J_{-2/3}(z_n) - J_{2/3}(z_n)]} \qquad (n > 1) \quad (2.18)$$

and the corresponding expansion of some function $w(\xi)$ in a series in the functions $\psi_n(\xi)$ (see [6] for a discussion of the expansibility conditions) is

$$w(\xi) = \sum_{n=1}^{\infty} w_n \psi_n(\xi), \quad w_n = \int_0^{\infty} w(\xi)\psi_n(\xi)\,d\xi \tag{2.19}$$

Let us use the above results to solve the problem formulated at the beginning of the present section. Denoting the functions occurring in the right sides of Eqs. (2.1) and (2.3) by $p(\xi, t)$ and $r(t)$, respectively, for brevity, we obtain

$$\frac{\partial^2 v}{\partial \xi^2} - \frac{\gamma}{2}\xi v - \frac{\partial v}{\partial t} = p(\xi, t), \qquad v|_{\xi=0} = r(t) \tag{2.20}$$

Multiplying the first equation of (2.20) by $\psi_n(\xi)d\xi$ and integrating from 0 to ∞, we obtain

$$\int_0^{\infty} \psi_n \frac{\partial^2 v}{\partial \xi^2} d\xi - \frac{\gamma}{2}\int_0^{\infty} \xi\psi_n v\,d\xi - \frac{dv_n}{dt} = p_n(t) \tag{2.21}$$

$$v_n = v_n(t) = \int_0^{\infty} v\psi_n d\xi, \qquad p_n = p_n(t) = \int_0^{\infty} p\psi_n d\xi \tag{2.22}$$

Integrating the first term on the left side and (2.21) twice by parts, and then recalling the first equation of (2.20) and the condition $\psi_n(0) = \psi_n(\infty) = 0$, as well as the fact that ψ_n satisfies the equation

$$\psi_n'' - (^1/_2\gamma\xi - \lambda_n)\psi_n = 0$$

we arrive at the relation

$$dv_n/dt + \lambda_n v_n = \psi_n'(0)\,r(t) - p_n(t) \quad (n \geqslant 1) \tag{2.23}$$

Integrating this equation (whose right side is known), we find under the initial condition $v_n|_{t=0} = 0$ which follows from (2.22) where $v|_{t=0} = 0$ that

$$v_n(t) = \int_0^{t} e^{-\lambda_n(t-\tau)}\,[\psi_n'(0)\,r(\tau) - p_n(\tau)]\,d\tau \tag{2.24}$$

The formula

$$v = v(\xi, t) = \sum_{n=1}^{\infty} v_n(t)\psi_n(\xi) \tag{2.25}$$

now gives us the solution of our problem.

We have considered in detail the case $\alpha < 0$ corresponding to a point spectrum. In the case $\alpha > 0$ when the spectrum becomes discontinuous will not be considered here (we refer the reader to the aforementioned monograph [6], Sect. 4.13).

3. Let us note some cases of solvability in known functions of one-, two-, and three-dimensional problems of heat conduction theory and similar problems in the case of moving boundaries.

Let us suppose that we are required to solve the problem formulated by the equation

$$\Delta u - \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial t} = f(x, y, z, t) \tag{3.1}$$

for a domain $x > R_1(t)$, unbounded in the direction of the positive x-axis whose cross section does not depend on x but can vary with time, and which we agree to call a "rod" although it may, in fact, be a plate. The boundary conditions for u can be assumed given for $x = R_1(t)$ and for $x = \infty$, as well as at the side surface of the rod; the initial value $u|_{t=0}$ can be set equal to zero, which does not limit the generality of the solution (by virtue of what was said in Sect. 1). We shall set $f \equiv 0$, which also does not limit the generality of the solution, since, if the eigenfunctions of the homogeneous problem are known, then these functions can be used to expand the solution of the nonhomogeneous problem.

Applying the transformation described in Sect. 1 to (3.1), i.e.

$$\xi = x - R_1(t), \qquad u = q_1 v, \qquad q_1 = \exp\left[-\frac{1}{2}\left(R_1'\xi + \frac{1}{2}\int R_1'^2 dt\right)\right] \quad (3.2)$$

we obtain

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{2} R_1'' \xi v - \frac{\partial v}{\partial t} = 0 \quad (3.3)$$

Let $R_1 = {}^1/_2\, \alpha t^2/2 + \beta t$. Equation (3.3) then becomes

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\alpha}{2} \xi v - \frac{\partial v}{\partial t} = 0 \quad (3.4)$$

Setting

$$v = \mu(\xi) w(y, z, t) \quad (3.5)$$

where $\mu(\xi)$ is one of the eigenfunctions of the problem considered in Sect. 2 which satisfies Eq. (3.5) and where λ is the corresponding eigenvalue (*), we find from (3.4) that

$$\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \frac{\alpha\lambda}{2} w - \frac{\partial w}{\partial t} = 0 \quad (3.6)$$

If the cross section of the domain is a rectangle with any ratio of sides (e.g. a strip of constant width), a disk, a circular ring, a sector, etc., whose dimensions do not vary with time, then, since Eq. (3.6) is amenable to separation of variables in Cartesian and polar coordinate systems (the latter in the plane yz, of course), general equation (3.1) for a domain with a cross section of similar shape (with the left hand boundary moving at constant velocity or constant acceleration) is also solvable in known functions.

In the case where the cross section varies with time we can obtain certain classes of problems solvable in known functions by combining a transformation of the type (3.2) (not only for the x-coordinate, but for other Cartesian coordinates as well) with the transformations described in [2, 3]. Thus, replacing the y and z in (3.6) by the independent variables η and ζ and introducing the new function P by means of the relations

$$\eta = \frac{y}{R_2}, \qquad \zeta = \frac{z}{R_3}, \qquad R_i = R_i(t) = \sqrt{A_i t^2 + 2B_i t + C_i} \quad (i = 2, 3) \quad (3.7)$$

(A_i, B_i, C_i are arbitrary constants) and

$$P = \frac{w}{Q}, \qquad Q = \frac{1}{\sqrt{R_2 R_3}} \exp\left[-\frac{R_2 R_2' \eta^2 + R_3 R_3' \zeta^2}{4}\right] \quad (3.8)$$

we obtain the following equation for P:

$$\frac{1}{R_2^2}\left[\frac{\partial^2 P}{\partial \eta^2} + \frac{A_2 C_2 - B_2^2}{4} \eta^2 P\right] + \frac{1}{R_3^2}\left[\frac{\partial^2 P}{\partial \zeta^2} + \frac{A_3 C_3 - B_3^2}{4} \zeta^2 P\right] + \\ + \frac{\alpha\lambda}{2} P - \frac{\partial P}{\partial t} = 0 \quad (3.9)$$

The variables in this equation become separable if we set

$$P = \Phi(\eta)\chi(\eta)\theta(t) \quad (3.10)$$

and impose the following conditions on the functions $\Phi(\eta)$ and $\chi(\zeta)$:

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*) For example, for $\alpha < 0$ the function $\mu(\xi)$ is one of the functions $\psi_n(\xi)$ defined by formula (2.18) and λ is the corresponding eigenvalue λ_n.

$$\frac{d^2\varphi}{d\eta^2}+\left[\frac{A_2C_2-B_2^2}{4}\eta^2+\nu_2\right]\varphi=0 \qquad (\nu_2=\text{const}) \tag{3.11}$$

$$\frac{d^2\chi}{d\zeta^2}+\left[\frac{A_3C_3-B_3^2}{4}\zeta^2+\nu_3\right]\chi=0 \qquad (\nu_3=\text{const}) \tag{3.12}$$

We then obtain the equation

$$\frac{d\theta}{dt}=\left(\frac{\alpha\lambda}{2}-\frac{\nu_2}{R_2^2}-\frac{\nu_3}{R_3^2}\right)\theta \tag{3.13}$$

for $\theta(t)$.

The foregoing implies that if the cross section of the domain is a rectangle whose sides perpendicular to the y- and z-axes move apart or move together with time according to laws of the form (3.7) with $i=2$ and $i=3$, respectively, and if the boundary values of the relative coordinates η and ζ do not depend on time, then we can expand the solution of the problem in the corresponding eigenfunctions of Eqs. (3.11), (3.12) corresponding to these (constant!) boundary values of η and ζ (see [2] for more details).

If $R_2(t)=R_3(t)$ (homogeneous expansion or compression), then Eq. (3.11) becomes simpler, i.e.,

$$\frac{1}{R_2^2}\left[\frac{\partial^2P}{\partial\eta^2}+\frac{\partial^2P}{\partial\zeta^2}+\frac{A_2C_2-B_2^2}{4}(\eta^2+\zeta^2)P\right]+\frac{\alpha\lambda}{2}P-\frac{\partial P}{\partial t}=0 \tag{3.14}$$

This form of the equation enables us to separate variables not only in the Cartesian coordinates η, ζ, but also in the corresponding polar coordinates. The problem then becomes solvable in known functions for disk, circular ring, and other cross sections.

In all of these cases the solution can be obtained according to the procedure described in detail in [2, 3].

4. We investigated the solution of boundary value problems for Eq. (3.1) in the case of a semi-infinite prismatic or cylindrical domain (in particular, for a half-space) under the assumption that the endface $x=R_1(t)$ moves according to law (3.2), and that the side faces move according to laws of the form (3.7).

A similar technique can be used to investigate any other combination of motions of these types along the axes x, y, z. For example, we can assume that the motion along the x-axis is described by the law

$$x=R_1(t)=\sqrt{A_1t^2+2B_1t+C_1} \quad (A_1,\ B_1,\ C_1=\text{const}) \tag{4.1}$$

and the motion along the y- and z-axes by law (3.7). This brings us to a particular case of the analogous problem for a domain in the shape of a rectangular parallelepiped with an arbitrary rib ratio at the initial instant whose faces move according to laws (3.7) and (4.1). In this problem (the general method for its solution is described in [3]) the length of the rib parallel to the x-axis must be set equal to infinity, which requires the introduction of eigenfunctions of an equation of the form

$$d^2u\,/\,d\xi^2+(\lambda-\alpha\xi^2)u=0 \tag{4.2}$$

over a semi-infinite segment.

Here α is a given constant and λ an arbitrary parameter. Considering the one-dimensional problem under the assumption that nothing depends on y and z, we obtain the solution of the initial problem for Eq. (1.1) formulated in Sect. 2 in the case where the motion of the half-space boundary is described by law (4.1).

In conclusion we note that introducing the new independent variables

$$\xi = x - R_1(t), \quad \eta = y - R_2(t), \quad \zeta = z - R_3(t) \tag{4.3}$$

into Eq. (3.1) and setting

$$\mathbf{R} = \mathbf{i}_x R_1 + \mathbf{i}_y R_2 + \mathbf{i}_z R_3, \quad \rho = \mathbf{i}_x \xi + \mathbf{i}_y \eta + \mathbf{i}_z \zeta \tag{4.4}$$

$$u = qv, \quad q = \exp[-{}^1/_2(\mathbf{R}\rho) + {}^1/_2(R\rho) + {}^1/_2 \int R'^2 dt]$$
$$R'^2 = R'^2_1 + R_2'^2 + R_3'^2 \tag{4.5}$$

where $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ are unit vectors along the coordinate axes, yields the generalization of Eq. (1.7) for the three-dimensional case,

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \zeta^2} + \frac{1}{2}(R_1''\xi + R_2''\eta + R_3''\zeta)v - \frac{\partial v}{\partial t} = -\frac{f(\xi + R_1, \eta + R_2, \zeta + R_3, t)}{q} \tag{4.6}$$

Considering a domain in the shape of a rectangular parallelepiped whose translational motion along the coordinate axes in space is described by formulas (4.3), so that the coordinates ξ, η, ζ of its faces retain time-independent constant values, and setting

$$R_i(t) = {}^1/_2 \alpha_i t^2 + \beta_i t + \gamma_i \qquad (\alpha_i, \beta_i, \gamma_i = \text{const}) \quad (i = 1,2,3) \tag{4.7}$$

we find that the variables in the homogeneous equation which follows from (4.6) for $f \equiv 0$ become separable and again obtain equations of the form (2.5) for the eigenfunctions along each of the coordinates ξ, η, ζ. These equations are valid in a finite or a semi-infinite interval depending on whether the corresponding rib of the parallelepiped is finite or infinite.

We have investigated problems whose solutions are expressible in terms of known functions, which is possible when the functions R, R_i, etc. are of a specific form. If these functions in Eqs. (1.7) or (4.6) have a form different from (3.7) or (4.7), then we can use the procedure described in Sect. 3 of [2] to reduce solution of the corresponding boundary value problems to relatively simple integral equations for the function v.

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THE SOLUTION OF ONE CLASS OF DUAL INTEGRAL EQUATIONS CONNECTED WITH THE MEHLER-FOCK TRANSFORM IN THE THEORY OF ELASTICITY AND MATHEMATICAL PHYSICS

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Dual integral equations with kernels containing spherical Legendre functions are examined. It is shown that these equations permit exact solution in quadratures. The proposed theory includes as a special case the theory of equations examined earlier which are connected with the Mehler-Fock transform and which are encountered in various applications, in particular in the solution of mixed boundary value problems in mathematical physics and in the theory of elasticity.

1. Equations of the following form are called dual equations connected with the integral transform of Mehler-Fock:

$$\int_0^\infty M(\tau) P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = f(\alpha) \qquad (0 \leqslant \alpha < \alpha_0)$$

$$\int_0^\infty M(\tau)\,\omega(\tau) P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = g(\alpha) \qquad (\alpha > \alpha_0) \tag{1.1}$$

here $P_\nu(z)$ is a spherical Legendre function with a complex index $\nu = -1/2 + i\tau$, $f(\alpha)$ and $g(\alpha)$ are given functions, $\omega(\tau)$ is the weight function ($\omega(\tau) > 0$, $\omega(\tau) \approx \tau$ for $\tau \to \infty$). Equations of this type are encountered in many applications; in particular, they play an important role in the solution of some mixed boundary value problems. Generalizations of Eqs. (1.1) are also examined. The kernels of these equations contain associated spherical functions.

At the present time a general theory of such equations does not exist, and a large part of results obtained in this area is related to equations of a special form which correspond to different selection of function $\omega(\tau)$ (see [1−6]). Thus, the following equations were studied

$$\int_0^\infty M(\tau) P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = f(\alpha) \qquad (0 \leqslant \alpha < \alpha_0) \tag{1.2}$$

$$\int_0^\infty M(\tau) \frac{\operatorname{ch}\pi\tau}{\pi [P_{-1/2+i\tau}(0)]^2} P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = 0 \qquad (\alpha > \alpha_0)$$